

FINITE-DIMENSIONAL REPRESENTATIONS CONSTRUCTED FROM RANDOM WALKS

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ABSTRACT. Given a 1-cocycle b with coefficients in an orthogonal representation, we show that any finite dimensional summand of b is cohomologically trivial if and only if $\|b(X_n)\|^2/n$ tends to a constant in probability, where X_n is the trajectory of the random walk (G, μ) . As a corollary, we obtain sufficient conditions for G to satisfy Shalom's property H_{FD} . Another application is a convergence to a constant in probability of $\mu^{*n}(e) - \mu^{*n}(g)$, $n \gg m$, normalized by its average with respect to μ^{*m} , for any amenable group without infinite virtually Abelian quotients. Finally, we show that the harmonic equivariant mapping of G to a Hilbert space obtained as an U -ultralimit of normalized $\mu^{*n} - g\mu^{*n}$ can depend on the ultrafilter U for some groups.

1. INTRODUCTION

Convention. Throughout the paper, G is a compactly generated locally compact group with a distinguished relatively compact symmetric subset Q which contains an open generating neighborhood e of G , and μ is a symmetric probability measure on G that satisfies the following conditions:

- μ is absolutely continuous with respect to the Haar measure m ,
- $\inf\{\frac{d\mu}{dm}(x) : x \in Q\} > 0$,
- $\int |x|_G^d d\mu(x) < \infty$ for all d .

Here $|x|_G := \min\{n : x \in Q^n\}$ (except that $|e|_G := 0$). Note that $|\cdot|_G$ is a pseudo-length, i.e., it satisfies $|x|_G = |x^{-1}|_G$ and $|xy|_G \leq |x|_G + |y|_G$. Put $B_G(r) := \{x \in G : |x|_G \leq r\}$.

Formulation of the results. Throughout the paper, we will work with real Hilbert spaces and orthogonal representations. This is purely for our convenience and all results (but not the proofs) hold true for complex Hilbert spaces and unitary representations (except that the statement of Theorem 2.4 has to be slightly modified), because any complex Hilbert space $\mathcal{H}_{\mathbb{C}}$ is also a real Hilbert space with the real inner product $(v, w) \mapsto \Re\langle v, w \rangle_{\mathcal{H}_{\mathbb{C}}}$, and any 1-cocycle (defined below) with coefficients in a unitary representation can be regarded as the one with coefficients in an orthogonal representation.

Let $\pi : G \curvearrowright \mathcal{H}$ be an orthogonal representation on a real Hilbert space \mathcal{H} . Recall that a 1-cocycle (or simply a cocycle) is a continuous map $b : G \rightarrow \mathcal{H}$ which satisfies the 1-cocycle identity: $b(gx) = b(g) + \pi_g b(x)$ for all $g, x \in G$. It is a 1-coboundary if there is $v \in \mathcal{H}$ such that $b(x) = v - \pi_x v$ for all $x \in G$. We note that b is a 1-coboundary if and only if it is bounded on G (Proposition 2.2.9 in [2]). Every cocycle b satisfies that $b(e) = 0$ and $\|b(x) - b(y)\| = \|b(x^{-1}y)\| \leq \|b\|_Q |x^{-1}y|_G$, where $\|b\|_Q := \sup_{g \in Q} \|b(g)\| < \infty$.

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A cocycle b is said to be μ -harmonic (or simply harmonic) if $\int b(gx) d\mu(x) = b(g)$ for all g , or equivalently $\int b(x) d\mu(x) = 0$. Any cocycle b gives rise to an affine isometric action $A: G \times \mathcal{H} \rightarrow \mathcal{H}$ by $A(g, v) = \pi_g v + b(g)$ (see Chapter 2 in [2]). Conversely, for any (affine) isometric action on a Hilbert space and a point $v \in \mathcal{H}$, the map $b(g) = A(g, v) - v$ defines a 1-cocycle, and harmonicity of this cocycle is same as harmonicity of the orbit map $g \mapsto A(g, v)$. Under an appropriate assumption on the decay of a non-degenerate measure μ , it is known that a compactly generated locally compact group G admits a non-zero μ -harmonic cocycle with respect to some orthogonal representation if and only if G does not satisfy Kazhdan' property (T). Existence of a non-zero harmonic cocycle on group which does not satisfy property (T) is proved by Mok ([24, Cor. 0.1]), Korevaar and Schoen [21, Thm 4.1.2] for finitely presented groups (and not discrete definition of harmonicity) and in general case (and discrete definition of harmonicity) by Shalom in [31]. See also Gromov [14], Fisher and Margulis [11], Lee and Peres [22, Thm 3.8], Ozawa [28] as well as the book by Bekka, de la Harpe, and Valette [2] for a non-exhaustive list of references about this result.

We say that a 1-cocycle b is *finite-dimensional* if the $\pi(G)$ -invariant subspace $\overline{\text{span}} b(G)$ is finite-dimensional. If $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ is the orthogonal decomposition of \mathcal{H} into $\pi(G)$ -invariant subspaces, then $b = \bigoplus_i P_{\mathcal{H}_i} b$ is a decomposition of b into 1-cocycles $P_{\mathcal{H}_i} b$ (with respect to $\pi|_{\mathcal{H}_i}$). We call each $P_{\mathcal{H}_i} b$ a *summand* of b . We say that such summand is *cohomologically trivial* if it is a 1-coboundary.

Given a probability measure μ on G , let X_n denote its trajectory of the random walk (G, μ) , that is, $X_n = s_1 s_2 \cdots s_n$ where increments $s_i \in G$ are independent and chosen with respect to μ . The corresponding probability measure and its expectation are denoted by \mathbb{P} and \mathbb{E} .

The value of a Hilbert valued μ -harmonic 1-cocycle along a trajectory of the random walk (G, μ) is a martingale, and therefore

$$\mathbb{E}[\|b(X_n)\|^2] = \sum_{k=1}^n \mathbb{E}[\|b(X_k)\|^2 - \|b(X_{k-1})\|^2] = n \mathbb{E}[\|b(X_1)\|^2].$$

That is, the expected value $\frac{1}{n} \mathbb{E}[\|b(X_n)\|^2]$ is equal to a constant, not depending on n . For any (not necessarily harmonic) 1-cocycle b , the expected value $\frac{1}{n} \mathbb{E}[\|b(X_n)\|^2]$ has a limit (see Lemma 2.2). Theorem A below characterizes the case when the random variable $\frac{1}{n} \|b(X_n)\|^2$ tends to a constant.

Theorem A. *Let G be a compactly generated locally compact group with a probability measure μ on G as in Convention. Let $b: G \rightarrow \mathcal{H}$ be a 1-cocycle. Then the following conditions are equivalent:*

- (1) *Any finite-dimensional summand of b is cohomologically trivial.*
- (2) *$\frac{1}{n} \|b(X_n)\|^2$ tends to a constant in probability.*

Now assume moreover that b is harmonic and put $c = \int_G \|b(x)\|^2 d\mu(x)$. Then the limit

$$\beta := \lim_{n \rightarrow \infty} \frac{1}{2c^2} \mathbb{E}\left[\left|\frac{\|b(X_n)\|^2}{n} - c\right|^2\right]$$

always exists, and $\beta = 0$ if and only if (1) and (2) hold. If $\beta \neq 0$, then b has a cohomologically non-trivial finite-dimensional summand of dimension $\leq 1/\beta$.

A more precise version of Theorem A will be given in Theorem 2.4, where we describe the limit distribution of $\|b(X_n)\|/\sqrt{n}$. This theorem has the following corollary:

Corollary. *Let b be a harmonic cocycle. Then, b is a direct sum of (possibly infinitely many) finite-dimensional cocycles if and only if $\limsup_n \mathbb{P}(\|b(X_n)\| < c\sqrt{n}) > 0$ for every $c > 0$.*

Recall that a group G is said to have *Shalom's property H_{FD}* if every orthogonal representation π with non-zero reduced cohomology group $\overline{H}^1(G, \pi)$ contains a non-zero finite-dimensional subrepresentation. In Corollary 2.5 we show that G satisfies Shalom's property H_{FD} if at least one of the two following conditions hold: either $\liminf_n \|\mu^{*n} - \mu^{*(1+\delta)n}\|_1 < 2$ for some $\delta > 0$ or $\limsup_n \mu^{*n}(B_G(c\sqrt{n})) > 0$ for all $c > 0$.

Theorem A and its corollaries develop the argument from [28]. While the main result of [28] is a new proof of Gromov's polynomial growth theorem, the paper also provides a more general criterion for the property H_{FD} for a finitely generated group in terms of convolutions of random walks is given in Section 4 of [28]. It is shown in [10] that wreath products of \mathbb{Z} with finite groups satisfy the assumption of that criterion, providing examples of groups of super-polynomial growth where the criterion applies. The assumption of the criterion from Section 4 in [28] uses shifted convolution, and it is not clear whether this assumption is defined by an unmarked Cayley graph of G . Assume that (G, μ) is a simple random walk on G , that is, μ is equidistributed on a finite generating set of G . The conditions of (1) as well as of (2) of Corollary 2.5 are clearly defined by the unmarked Cayley graph of G . We do not know any group which satisfies the assumption of (1) or of (2) of Corollary 2.5 and for which we know that it violates the assumption of Section 4 of [28]. But the conditions of Corollary 2.5 are easier to check than the assumption from [28]. For example, it is easily applicable to solvable Baumslag–Solitar groups, lamplighter groups $\mathbb{Z} \ltimes \bigoplus_{\mathbb{Z}} F$ with F finite, or to polycyclic groups obtained as extension of \mathbb{Z}^2 by $M \in SL(2, d)$ with eigenvalues of absolute value $\neq 1$. See Section 3 for more examples. We do not know any group which satisfies Shalom's property and does not satisfy the assumption of Corollary 2.5.

Given a not necessarily harmonic cocycle b on a group without property (T), a harmonic cocycle can be obtained taking averages of b (see Mok, Korevaar–Schoen, Shalom [21, 24, 30]), and in particular this can be achieved averaging with respect to a probability measure μ (see e.g. Gromov, Lee–Peres [14, 22]). In Section 4 we study the cocycle b_μ , constructed as a (ultra)-limit in $\ell_2(G)$ of normalized $\mu^{*n} - g\mu^{*n}$ on a finitely generated amenable group G . Kesten's criterion [20] (see also [1]) implies that μ^{*n} is a sequence of almost invariant vectors in $\ell_2(G)$, and one can moreover show (see Corollary 4.2) that the limit is a harmonic 1-cocycle. Applying Theorem A to this 1-cocycle, one obtains

Theorem B. *Let G be a finitely generated amenable group without virtually Abelian infinite quotients. Let μ be a symmetric non-degenerate measure with finite moments. Then $(\mu^{*2n}(e) - \mu^{*2n}(g))/\alpha(m, n)$ tends to a constant in probability μ^{*2m} as $m \rightarrow \infty$ and $n \gg m$. Here $\alpha(m, n) = \mu^{*2n}(e) - \mu^{*2n+2m}(e)$ is the average of $\mu^{*2n}(e) - \mu^{*2n}(g)$ with respect to μ^{*2m} . Namely*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left| \frac{\mu^{*2n}(e) - \mu^{*2n}(X_m)}{\mu^{*2n}(e) - \mu^{*2n+2m}(e)} - 1 \right| = 0.$$

Take n much larger than m . Observe that a group is amenable if and only if $\mu^{*2n}(g)/\mu^{*2n}(e)$ is close to 1 in probability with respect to μ^{*2m} . Theorem B gives a sufficient condition for the concentration of the second order term of μ^{*2n} .

Theorem B applies in particular to any finitely generated amenable torsion group (such as Grigorchuk groups G_w [12]) or to any finitely generated amenable simple group (such as commutator full topological groups of minimal shifts on \mathbb{Z} (which are simple Matui [23] and

amenable Juschenko–Monod [18]), or to simple groups of intermediate growth constructed recently by Nekrashevych [25]. If μ is equidistributed on a finite generated set of G , then the assumption of Theorem B depends only on the unmarked Cayley graph of (G, μ) . In particular, the theorem gives a necessary condition for an amenable group to be simple in terms of unmarked Cayley graphs. In general, it is known that the property of being simple can not be defined by the unmarked Cayley graphs, as it is shown by Burger and Mozes [4] (their examples are isometric to product of two trees and they are non-amenable). It is to our knowledge an open problem whether a property of being a torsion group can be verified geometrically.

Geometric group theory tries to recover properties of a group from the word metrics of this group. Given a group G , generated by a finite set S , its action on a metric space X and a point $x_0 \in X$, the group G is equipped with two metrics: the word metric $d_{G,S}(g, h)$ as well as $d_{X,x_0}(g, h) = d_X(gx_0, hx_0)$. It seems interesting to study which properties of the action, or of the group G , can be recovered from these two metrics. Theorem A as well as Corollary 2.5 provide examples of such situation, for X being a Hilbert space and a group G acting by affine transformations of X .

Fix a non-principal ultrafilter U . Let $b_{U,\mu,G}^{p,q}$ be the mapping to a vector space equipped with a metric, constructed as U ultralimit of normalized $g(\mu^{*n})^q - (\mu^{*n})^q$, considered as elements of $\ell_p(G)$. We recall that any ultralimit of Hilbert spaces is a Hilbert space, so that for $p = 2$ and any $q \geq 0$ we obtain a cocycle with respect to some orthogonal representation of \mathcal{H} . In particular, for $q = 1$ and $p = 2$, $b_{U,\mu,G}^{p,q}$ coincides up to a multiplicative constant with the harmonic cocycle $b_{U,\mu}$, studied in the proof of Theorem B in Section 4. In general, for $p \neq 2$, we obtain a cocycle with respect to some metric preserving representation on a linear space, which is an ultralimit of ℓ_p .

In Theorem C below we show that the cocycles $b_{\mu,U}^{p,q}$, $p \geq 1$, $q \geq 0$ (in particular, the harmonic cocycle $b_{\mu,U}$) can depend on the choice of a non-principal ultrafilter.

Theorem C. *Take $p = 1$ or 2 and $q = 0, 1$, or 2 . For any $D \geq 2$ there exists torsion groups G_1, G_2, \dots, G_D such that the following holds. Consider finitely supported symmetric non-degenerate measures μ_i on G_i and put $G = \prod_{i=1}^D G_i$, $\mu = \prod_{i=1}^D \mu_i$. For each $j : 1 \leq j \leq D$ there exists a non-principal ultrafilter U such that the limiting cocycle is a limit along some subsequence n_i and it satisfies $b_{\mu,G}^{p,q} = b_{\mu_j,G_j}^{p,q}$.*

Theorem C shows in particular that there exist at least D not equal limiting cocycles among $b_{\mu,G}^{p,q}$, and at least D not equal subgroups among possible kernels of such cocycles. Such groups G admit $g_1, g_2 \in G$ such that the ratio $(\mu^{*2n}(e) - \mu^{*2n}(g_1))/(\mu^{*2n}(e) - \mu^{*2n}(g_2))$ does not have a limit as $n \rightarrow \infty$.

The groups G_i are constructed as piecewise automatic groups [9], they can be chosen to be of sub-exponential word growth, but in such a way that for each j the group G_j is in some sense very close to a non-amenable group on some scale while on this particular scale it does not happen to other G_k , $j \neq k$. The contribution to $b_{\mu,G}^{p,q}$ is mainly from G_j on this scale, and the kernel of $b_{\mu,G}^{p,q}$ contains $\prod_{k \neq j} G_k$.

The kernels of cocycles $b_{\mu,G}^{p,q}$ are particular cases of what we call ℓ_p -thin subgroups: this is a natural family of subgroups, related to the shifts $(\mu^{*n})^q$ (see Definition 5.1), which for $p = 2$ is related to amenability, for $p = 1$ to Poisson-Furstenberg boundary and for $p = 0$ to growth of groups (see Lemma 5.5), these groups in some situation may depend on p (see Example 5.8) and on the measure a finitely supported measure μ (see Remark 5.9).

Since the group G in the statement of the theorem is a torsion group, it does not admit virtual quotient to an infinite cyclic group. In particular, taking $p = 2$ we can apply the conclusion Theorem B to (G, μ) to claim that $\mu^{*n}(e) - \mu^{*n}(g)$, normalised by its average $\alpha(m, n)$ is close to a constant in probability μ^{*m} , for $n \gg m$. In other words, for each $n \gg m$ μ^{*m} is concentrated on a set where normalized $\mu^{*n}(e) - \mu^{*n}(g)$ is close to its mean value, but in view of C these sets may depend essentially on n .

2. HARMONIC COCYCLES AND FINITE-DIMENSIONAL SUMMANDS

We now recall from Sections 4 and 5 in [15] that the space $Z^1(G, \pi)$ of 1-cocycles is a Hilbert space under the norm

$$\|b\|_{L^2(\mu)} := \left(\int_G \|b(x)\|^2 d\mu(x) \right)^{1/2},$$

and it decomposes into an orthogonal direct sum of approximate 1-coboundaries and μ -harmonic 1-cocycles. We will say b is *normalized* when $\|b\|_{L^2(\mu)} = 1$.

Lemma 2.1. *The space $Z^1(G, \pi)$ of 1-cocycles is a Hilbert space with respect to the norm $\|\cdot\|_{L^2(\mu)}$. Moreover the norms $\|\cdot\|_{L^2(\mu)}$ and $\|\cdot\|_Q$ are equivalent.*

Proof. We observe that $Z^1(G, \pi)$ is a Banach space w.r.t. the norm $\|\cdot\|_Q$ (see [2, Chapter 3]), and that $\|b\|_{L^2(\mu)} \leq (\int_G |x|_G^2 d\mu(x))^{1/2} \|b\|_Q$. The other side inequality follows, via Open Mapping Theorem, from the fact that any measurable locally integrable 1-cocycle into a separable Hilbert space is automatically continuous modulo a null set. However, following [15], we give a more direct proof here. Take an open generating neighborhood U of e such that $U \subset Q$ and an open neighborhood V of e such that $V^2 \subset U$. We observe that $(\int \|b(x)\|^2 d\mu^{*2}(x))^{1/2} \leq 2\|b\|_{L^2(\mu)}$ and $\varepsilon := \inf_{x \in UV} \frac{d\mu^{*2}}{d\mu}(x) > 0$. Thus, for every $g \in U$ one has

$$\begin{aligned} \|b(g)\|^2 &= m(V)^{-1} \int_V \|b(gx) - \pi_g b(x)\|^2 dm(x) \\ &\leq 2m(V)^{-1} \left[\int_{gV} \|b(x)\|^2 dm(x) + \int_V \|b(x)\|^2 dm(x) \right] \\ &\leq 4\varepsilon^{-1} m(V)^{-1} \int_{UV} \|b(x)\|^2 d\mu^{*2}(x) \\ &\leq 16\varepsilon^{-1} m(V)^{-1} \|b\|_{L^2(\mu)}^2. \end{aligned}$$

Since there is $N \in \mathbb{N}$ such that $Q \subset U^N$, this proves that the norms $\|\cdot\|_{L^2(\mu)}$ and $\|\cdot\|_Q$ are equivalent, and that $Z^1(G, \pi)$ is a Hilbert space w.r.t. the norm $\|\cdot\|_{L^2(\mu)}$. \square

The *reduced 1-cohomology group* $\overline{H}^1(G, \pi) := Z^1(G, \pi) / \overline{B^1(G, \pi)}$ is defined to be the space $Z^1(G, \pi)$ of 1-cocycles modulo the closure of the subspace $B^1(G, \pi)$ of 1-coboundaries. We note that $\overline{B^1(G, \pi)} = B^1(G, \pi)$ if π is finite-dimensional, by Theorem 1 in [15]. See Chapter 3 in [2] for an introduction to first reduced cohomology groups. Thus,

$$Z^1(G, \pi) = \overline{B^1(G, \pi)} \oplus B^1(G, \pi)^\perp \text{ and } \overline{H}^1(G, \pi) \cong B^1(G, \pi)^\perp.$$

We observe that $b \in Z^1(G, \pi)$ belongs to $B^1(G, \pi)^\perp$ if and only if it is μ -harmonic in the sense $\int b(x) d\mu(x) = 0$ or equivalently $\int b(gx) d\mu(x) = b(g)$ for all $g \in G$. Indeed, this follows from the identities $b(x^{-1}) + \pi_x^{-1} b(x) = b(e) = 0$ and

$$\int \langle b(x), v - \pi_x v \rangle d\mu(x) = 2 \langle \int b(x) d\mu(x), v \rangle.$$

We note that every summand of a μ -harmonic 1-cocycle is μ -harmonic and that every non-zero μ -harmonic 1-cocycle is not a 1-coboundary.

We recall the general fact about orthogonal representations. Let (π, \mathcal{H}) be an orthogonal representation of G and put

$$T_0 := \mathbb{E}[\pi(X_1)] = \int \pi(g) d\mu(g).$$

Then, T_0 is a self-adjoint contraction on the Hilbert space \mathcal{H} such that $T_0^k = \mathbb{E}[\pi(X_k)]$ for every k . By strict convexity of a Hilbert space, a vector $v \in \mathcal{H}$ satisfies $T_0 v = v$ if and only if $\pi_g v = v$ for μ -a.e. g , which is equivalent to that v is $\pi(G)$ -invariant. Thus by spectral theory, the operators $\frac{1}{n} \sum_{k=0}^{n-1} T_0^k = \mathbb{E}[\frac{1}{n} \sum_{k=0}^{n-1} \pi(X_k)]$ converge in strong operator topology to the orthogonal projection P_0 onto the subspace of $\pi(G)$ -invariant vectors. One moreover has convergence in probability

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \pi(X_k) v - P_0 v \right\| \xrightarrow{\mathbb{P}} 0.$$

Indeed, to prove it, one may assume $P_0 = 0$ and in this case

$$\mathbb{E} \left[\left\| \frac{1}{n} \sum_{k=0}^{n-1} \pi(X_k) v \right\|^2 \right] = \frac{1}{n^2} \sum_{k,l=0}^{n-1} \langle T_0^{|k-l|} v, v \rangle \rightarrow 0.$$

Lemma 2.2. *For every $b \in Z^1(G, \pi) = \overline{B^1(G, \pi)} \oplus B^1(G, \pi)^\perp$, one has*

$$\lim_n \frac{1}{n} \mathbb{E}[\|b(X_n)\|^2] = \|b_{\text{harm}}\|_{L^2(\mu)}^2,$$

where b_{harm} is the $B^1(G, \pi)^\perp$ summand in the above decomposition. In particular, b is nonzero in $\overline{B^1(G, \pi)}$ if and only if $\lim_n \frac{1}{n} \mathbb{E}[\|b(X_n)\|^2] > 0$.

Proof. Let $T_0 := \int \pi(g) d\mu(g)$. If $c \in B^1(\pi, \mathcal{H})$ is a 1-coboundary, $c(x) = v - \pi_x v$, then for every n one has

$$\frac{1}{n} \mathbb{E}[\|c(X_n)\|^2] = \frac{2}{n} \langle (1 - T_0^n) v, v \rangle \leq 2 \langle (1 - T_0) v, v \rangle = \|c\|_{L^2(\mu)}^2.$$

Since $c \mapsto \mathbb{E}[\|c(X_n)\|^2]$ is norm-continuous by Lemma 2.1, the above inequality holds for all $c \in \overline{B^1(G, \pi)}$. Hence, for any $c \in \overline{B^1(G, \pi)}$, by approximating it by $c_n \in B^1(G, \pi)$, one has

$$\limsup_n \frac{1}{n} \mathbb{E}[\|c(X_n)\|^2] = \limsup_n \frac{1}{n} \mathbb{E}[\|(c - c_n)(X_n)\|^2] \leq \|c - c_n\|_{L^2(\mu)}^2 \rightarrow 0.$$

Now let $b = c + b_{\text{harm}} \in \overline{B^1(G, \pi)} + B^1(G, \pi)^\perp$ be given. Note that since b_{harm} is μ^{*n} -harmonic, it is orthogonal to c in $L^2(\mu^{*n})$. Consequently, one has

$$\lim_n \frac{1}{n} \mathbb{E}[\|b(X_n)\|^2] = \lim_n \frac{1}{n} \mathbb{E}[\|c(X_n)\|^2 + \|b_{\text{harm}}(X_n)\|^2] = \|b_{\text{harm}}\|_{L^2(\mu)}^2. \quad \square$$

It is not clear whether $\frac{1}{n^2} \mathbb{E}[\|b(X_n)\|^4]$ is bounded for every 1-cocycle b . However, it is the case for any μ -harmonic 1-cocycle b (cf. Footnote 2 in [22]).

Lemma 2.3. *For every d , one has*

$$\sup_n \sup_b \frac{1}{n^d} \mathbb{E}[\|b(X_n)\|^{2d}] < \infty,$$

where the supremum runs over all normalized μ -harmonic 1-cocycles b .

Proof. We fix a universal orthogonal representation (π, \mathcal{H}) and consider the operators U_n from the space of μ -harmonic cocycles into $L^{2d}(\mu^{*n}; \mathcal{H})$, given by $U_n b = n^{-1/2}b$. Since

$$\begin{aligned} \|U_n b\| &= \left(\frac{1}{n^d} \mathbb{E}[\|b(X_n)\|^{2d}] \right)^{1/2d} \\ &\leq n^{1/2} \|b\|_{L^{2d}(\mu; \mathcal{H})} \leq n^{1/2} \|b\|_Q (\mathbb{E}[|X_1|_G^{2d}])^{1/2d}, \end{aligned}$$

the operators U_n are bounded by Lemma 2.1. The lemma claims that U_n 's are uniformly bounded. For this, by Principle of Uniform Boundedness, it suffices to show $\sup_n \|U_n b\| < \infty$ for each b . (The use of PUB can be avoided if one does the following proof more meticulously.) We in fact prove that $\limsup_n \frac{1}{n^d} \mathbb{E}[\|b(X_n)\|^{2d}] \leq (2d-1)!!$ for each harmonic cocycle b , by induction on d . The case $d = 1$ is clear. By induction hypothesis and the Cauchy–Schwarz inequality when k is odd, we may assume that $\mathbb{E}[\|b(X_n)\|^k] = O(n^{k/2})$ for all $k \leq 2(d-1)$. It follows that

$$\begin{aligned} \mathbb{E}[\|b(X_n)\|^{2d}] &= \iint \|b(x) - b(y)\|^{2d} d\mu^{*n-1}(x) d\mu(y) \\ &= \iint (\|b(x)\|^2 - 2\langle b(x), b(y) \rangle + \|b(y)\|^2)^d d\mu^{*n-1}(x) d\mu(y) \\ &= \iint \|b(x)\|^{2d} + \binom{d}{1} \|b(x)\|^{2(d-1)} \|b(y)\|^2 \\ &\quad + 4 \binom{d}{2} \|b(x)\|^{2(d-2)} |\langle b(x), b(y) \rangle|^2 d\mu^{*n-1}(x) d\mu(y) + O(n^{(2d-3)/2}) \\ &\leq \mathbb{E}[\|b(X_{n-1})\|^{2d}] + (d + 2d(d-1)) \cdot (2d-3)!! \cdot n^{d-1} + o(n^{d-1}) \\ &\leq \dots \leq (2d-1)!! \cdot n^d + o(n^d). \end{aligned}$$

This finishes the proof. \square

We start the proof of Theorem A. Recall that the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}$ is canonically identified with the space of Hilbert–Schmidt operators $\mathcal{S}_2(\mathcal{H})$ on \mathcal{H} via $v' \otimes v \leftrightarrow S_{v' \otimes v}$, where $S_{v' \otimes v}(u) = \langle u, v \rangle v'$. Under this identification, the operators $\pi_g \otimes \pi_g$ on $\mathcal{H} \otimes \mathcal{H}$ act on $\mathcal{S}_2(\mathcal{H})$ by conjugation $\text{Ad } \pi_g: S \mapsto \pi_g S \pi_g^*$. Every Hilbert–Schmidt operator is compact and every compact self-adjoint operator S has a unique spectral decomposition $S = \sum_i \lambda_i E_i$ where $\lambda_i \in \mathbb{R}$ are the non-zero eigenvalues of S and E_i are the finite-rank orthogonal projections onto the corresponding eigenspaces. If $v \in \mathcal{H} \otimes \mathcal{H}$ is $(\pi \otimes \pi)(G)$ -invariant, then S_v is $\text{Ad } \pi(G)$ -invariant and so are the spectral projections E_i 's, which means that $E_i \mathcal{H}$ are finite-dimensional $\pi(G)$ -invariant subspaces.

Now let us consider a 1-cocycle $b: G \rightarrow \mathcal{H}$ and put

$$w := \int (b \otimes b)(x) d\mu(x) \in \mathcal{H} \otimes \mathcal{H} \text{ and } T := \int \pi_g \otimes \pi_g d\mu(g).$$

Then, T is a self-adjoint contraction on $\mathcal{H} \otimes \mathcal{H}$, which is positivity preserving as an operator on $\mathcal{S}_2(\mathcal{H})$. By the previous discussion, $\frac{1}{n} \sum_{k=0}^{n-1} T^k$ converges in strong operator topology to the orthogonal projection P from $\mathcal{H} \otimes \mathcal{H}$ onto the subspace of $(\pi \otimes \pi)(G)$ -invariant vectors. In particular, $\frac{1}{n} \sum_{k=0}^{n-1} T^k w$ converges to Pw in norm and S_{Pw} is a positive Hilbert–Schmidt operator which is $\text{Ad } \pi(G)$ -invariant. If b is finite-dimensional, then the trace Tr is norm-continuous and

$$\text{Tr}(S_{Pw}) = \text{Tr}(\lim_n \frac{1}{n} \sum_{k=0}^{n-1} S_{T^k w}) = \text{Tr}(S_w) = \|b\|_{L^2(\mu)}^2.$$

In general, one has the spectral decomposition $S_{Pw} = \sum_i \lambda_i E_i$ where $\lambda_1, \lambda_2, \dots$ is a finite or infinite sequence of strictly positive numbers and $E_i \mathcal{H}$'s are finite-dimensional $\pi(G)$ -invariant subspaces. Thus for $b_i := E_i b$ and $b_\infty := b - (\sum_i b_i)$, one has the direct sum decomposition $b = b_\infty + \sum_i b_i$. We claim that each 1-cocycle b_i , $i \neq \infty$, is nonzero and that b_∞ does not admit a nonzero finite-dimensional summand anymore. This follows from the fact that $S_{Pw} = S_{Pw_\infty} \oplus \sum_i^\oplus S_{Pw_i}$ in accordance with $b = b_\infty \oplus \sum_i^\oplus b_i$ and hence $S_{Pw_i} = \lambda_i E_i$, where $w_i = \int (b_i \otimes b_i)(x) d\mu(x)$, including the case $i = \infty$ and $\lambda_\infty := 0$. Thus $\|Pw\| \neq 0$ if and only if b has a nonzero finite-dimensional summand. Moreover, one has

$$\mathrm{Tr}(S_{Pw}) = \sum_i \mathrm{Tr}(S_{Pw_i}) = \sum_i \lambda_i \mathrm{Tr}(E_i) = \sum_i \|b_i\|_{L^2(\mu)}^2$$

and

$$\|Pw\|^2 = \mathrm{Tr}(S_{Pw}^2) = \sum_i \lambda_i^2 \mathrm{Tr}(E_i).$$

For the proof of Theorem A, in view of Lemma 2.2 and the fact that any nonzero μ -harmonic 1-cocycle is cohomologically non-trivial, we may assume that the 1-cocycle b is μ -harmonic. For such b , we have the following more precise form of Theorem A.

Theorem 2.4. *Let G be as in Convention. Let b be a normalized μ -harmonic 1-cocycle and w , Pw , and $S_{Pw} = \sum_i \lambda_i E_i$ be as above. Then,*

$$\lim_{n \rightarrow \infty} \frac{1}{2} \mathbb{E} \left[\left| \frac{\|b(X_n)\|^2}{n} - 1 \right|^2 \right] = \|Pw\|^2 \leq (\min_i \dim E_i \mathcal{H})^{-1}.$$

Moreover, the random variables $\frac{1}{\sqrt{n}} \|b(X_n)\|$ converge in distribution and in moments to $(g_\infty^2 + \sum_{i,j} g_{i,j}^2)^{1/2}$, where $g_\infty^2 = \|b_\infty\|_{L^2(\mu)}^2$ is a constant function and $\{g_{i,j} : \lambda_i > 0, j = 1, \dots, \dim E_i \mathcal{H}\}$ is an independent family of centered Gaussian random variables of variances $\mathbb{E}[|g_{i,j}|^2] = \lambda_i$.

Proof of Theorem A and Theorem 2.4. Let b be a normalized μ -harmonic 1-cocycle. In the discussion above, we already saw $\|Pw\| \neq 0$ if and only if b has a nonzero finite-dimensional summand. Moreover the above formula implies

$$\|Pw\|^2 = \sum_i \lambda_i^2 \mathrm{Tr}(E_i) \leq (\max_i \lambda_i) \mathrm{Tr}(S_{Pw}) \leq (\min_i \mathrm{Tr}(E_i))^{-1},$$

since $\mathrm{Tr}(S_{Pw}) = \sum_i \lambda_i \mathrm{Tr}(E_i) \leq 1$. Note that $\mathrm{Tr}(E_i) = \dim E_i \mathcal{H}$.

Next, we prove that $\mathbb{E} \left[\left| \frac{\|b(X_n)\|^2}{n} - 1 \right|^2 \right] \rightarrow 2 \langle Pw, w \rangle = 2 \|Pw\|^2$. Recall that

$$\begin{aligned} \int (b \otimes b)(x) d\mu^{*n}(x) &= \iint (b \otimes b)(xy) d\mu^{*n-1}(x) d\mu(y) \\ &= \iint (b \otimes b)(x) + (\pi_x \otimes \pi_x)(b \otimes b)(y) d\mu^{*n-1}(x) d\mu(y) \\ &= \int (b \otimes b)(x) d\mu^{*n-1}(x) + T^{n-1}w \\ &= (1 + T + \dots + T^{n-1})w, \end{aligned}$$

and $\int \|b(x)\|^2 d\mu^{*n}(x) = n$ (see [22] and [28]). Hence

$$\begin{aligned}
\mathbb{E}[\|b(X_n)\|^4] &= \int \|b(x)\|^4 d\mu^{*n}(x) \\
&= \iint (\|b(x) - b(y)\|^2)^2 d\mu^{*n-1}(x) d\mu(y) \\
&= \iint [\|b(x)\|^4 + 4|\langle b(x), b(y) \rangle|^2 + \|b(y)\|^4 + 2\|b(x)\|^2 \|b(y)\|^2] d\mu^{*n-1}(x) d\mu(y) \\
&= \mathbb{E}[\|b(X_{n-1})\|^4] + 4\langle \sum_{k=0}^{n-2} T^k w, w \rangle + \mathbb{E}[\|b(X_1)\|^4] + 2(n-1) \\
&= 4\langle \sum_{k=1}^{n-1} (n-k) T^{k-1} w, w \rangle + n \mathbb{E}[\|b(X_1)\|^4] + n(n-1) \\
&\leq 3n^2 + O(n).
\end{aligned}$$

By Bounded Convergence Theorem, this implies that

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{\|b(X_n)\|^2}{n} - 1\right|^2\right] &= \mathbb{E}\left[\frac{1}{n^2} \|b(X_n)\|^4 - \frac{2}{n} \|b(X_n)\|^2 + 1\right] \\
&= \frac{4}{n^2} \langle \sum_{k=1}^{n-1} (n-k) T^{k-1} w, w \rangle + \frac{1}{n} (\mathbb{E}[\|b(X_1)\|^4] - 1) \\
&\rightarrow 2\langle Pw, w \rangle.
\end{aligned}$$

Now since $\sup_n \mathbb{E}\left[\left|\frac{1}{n} \|b(X_n)\|^2 - 1\right|^3\right] < \infty$ by Lemma 2.3, the sequence $\frac{1}{n} \|b(X_n)\|^2$ tends to a constant (which is necessarily 1) in probability if and only if one has $\mathbb{E}\left[\left|\frac{1}{n} \|b(X_n)\|^2 - 1\right|^2\right] \rightarrow 0$. This completes the proof of Theorem A and the first part of Theorem 2.4.

For the second half of Theorem 2.4, we first note that convergence in distribution and convergence in moments are equivalent in our setting. Indeed, by the moment condition (Lemma 2.3), convergence in distribution implies that in moments (see Corollary to 25.12 in [3]). The converse is also true when the limiting distribution is a (square sum of) normal distribution, by the Method of Moments (see Theorems 30.1 and 30.2 in [3]).

We use Martingale Central Limit Theorem (Theorem 35.12 in [3]) to prove that for any $v \in \mathcal{H}$ the random variables $S_n := n^{-1/2} \langle b(X_n), v \rangle$ converge to a normal distribution $N(0, q(v))$ where $q(v) = \langle S_{Pw} v, v \rangle$. Consider the martingale array $S_{n,k} := n^{-1/2} \langle b(X_k), v \rangle$, $k = 1, \dots, n$, and put

$$Y_{n,k} := S_{n,k} - S_{n,k-1} = n^{-1/2} \langle \pi(X_{k-1}) b(X_{k-1}^{-1} X_k), v \rangle.$$

Since $X_{k-1}^{-1} X_k$ has the same distribution as X_1 , one has

$$\begin{aligned}
\sum_{k=1}^n \mathbb{E}[Y_{n,k}^2 \mid X_1, \dots, X_{k-1}] &= \frac{1}{n} \sum_{k=1}^n \langle (\pi \otimes \pi)(X_{k-1}) w, v \otimes v \rangle \\
&\xrightarrow{\mathbb{P}} \langle Pw, v \otimes v \rangle = q(v),
\end{aligned}$$

and, for every ε ,

$$\sum_{k=1}^n \mathbb{E}[Y_{n,k}^2 1_{\{|Y_{n,k}| \geq \varepsilon\}}] \leq \mathbb{E}[\|b(X_1)\|^2 \|v\|^2 1_{\{\|b(X_1)\| \geq \varepsilon n^{1/2}\}}] \rightarrow 0.$$

Hence by Martingale CLT, one has $S_{n,n} \Rightarrow N(0, q(v))$ in distribution.

Now recall that $S_{Pw} = \sum_i \lambda_i E_i$ and $b = b_\infty + \sum b_i$, and take an orthonormal basis $\{v_{i,j} : j = 1, \dots, \text{Tr}(E_i)\}$ of $E_i \mathcal{H}$. Then, by the previous paragraph, $n^{-1/2} \langle b(X_n), v_{i,j} \rangle$ converges in distribution to a centered Gaussian random variable $g_{i,j}$ with variance $q(v_{i,j}) = \lambda_i$. Moreover, for any $\beta_{i,j} \in \mathbb{R}$, the random variables $\sum_{i,j} \beta_{i,j} n^{-1/2} \langle b(X_n), v_{i,j} \rangle$ converge in moments to $N(0, q(\sum_{i,j} \beta_{i,j} v_{i,j}))$, where

$$q(\sum_{i,j} \beta_{i,j} v_{i,j}) = \sum_{i,j} \beta_{i,j}^2 \lambda_i = \sum_{i,j} \beta_{i,j}^2 q(v_{i,j}).$$

This means that the family $\{\langle n^{-1/2} b(X_n), v_{i,j} \rangle\}_{i,j}$ are asymptotically independent as $n \rightarrow \infty$. Thus, for any $k \in \mathbb{N}$, one has

$$\frac{1}{n} \left\| \sum_{i=1}^k b_i(X_n) \right\|^2 = \sum_{i=1}^k \sum_j |n^{-1/2} \langle b(X_n), v_{i,j} \rangle|^2 \Rightarrow \sum_{i=1}^k \sum_j g_{i,j}^2,$$

where $g_{i,j}$ are independent centered Gaussian random variables with variances $\lambda_{i,j}$. Since

$$\limsup_k \mathbb{E} \left[\left(\frac{1}{n} \left\| \sum_{i=1}^k b_i(X_n) \right\|^2 \right)^d \right] \leq \lim_k C_d \left\| \sum_{i=1}^k b_i \right\|_{L^2(\mu)}^{2d} = 0$$

where C_d is a constant independent of k (by Lemma 2.3), one has

$$\lim_n \mathbb{E} \left[\left(\frac{1}{n} \left\| \sum_{i=1}^k b_i(X_n) \right\|^2 \right)^d \right] = \lim_k \lim_n \mathbb{E} \left[\left(\frac{1}{n} \left\| \sum_{i=1}^k b_i(X_n) \right\|^2 \right)^d \right]$$

for every d . Also, since $\frac{1}{n} \|b_\infty(X_n)\|^2 \rightarrow \|b_\infty\|_{L^2(\mu)}^2$ in moments by the first half of the proof, one has $\frac{1}{n} \|b(X_n)\|^2 \rightarrow g_\infty^2 + \sum_{i,j} g_{i,j}^2$ in moments. \square

Recall that a group G is said to have *Shalom's property* H_{FD} ([30]) if every orthogonal representation π with $\overline{H^1}(G, \pi) \neq 0$ contains a non-zero finite-dimensional subrepresentation. In other words, G has property H_{FD} if and only if every μ -harmonic 1-cocycle b decomposes into a (possibly infinite) direct sum of finite-dimensional summands. By Theorem 2.4, the latter happens for b if and only if $\lim_n \mu^{*n}(\{x \in G : \|b(x)\| \leq c\sqrt{n}\}) > 0$ for all $c > 0$.

Corollary 2.5. *Assume either (1) $\liminf_n \|\mu^{*n} - \mu^{*(1+\delta)n}\|_1 < 2$ for some $\delta > 0$ or (2) $\limsup_n \mu^{*n}(B_G(c\sqrt{n})) > 0$ for all $c > 0$. Then, G has Shalom's property H_{FD} .*

Proof. We prove a stronger statement that if G does not have property H_{FD} , then for every $\delta > 0$ there are $c > 0$ and a sequence $(E_n)_n$ of open subsets in G such that $\mu^{*n}(E_n) \rightarrow 1$ and $\mu^{*(1+\delta)n}(B_G(c\sqrt{n})E_n B_G(c\sqrt{n})) \rightarrow 0$.

Suppose that there is μ -harmonic 1-cocycle $b: G \rightarrow \mathcal{H}$ without a non-zero finite-dimensional summand. We can assume that this cocycle is normalized. Take any $0 < \delta < 1$. Put $c := (20\|b\|_Q)^{-1}\delta$ and

$$E_n := \{x \in G : \|b(x)\|^2 < (1 + \delta/4)n\}.$$

Then, for every $x \in E_n$ and $y, z \in B_G(c\sqrt{n})$ one has

$$\|b(yxz)\|^2 \leq \|b(x)\|^2 + 2\|b(x)\|\|b(y) + \pi_{yx}b(z)\| + \|b(y) + \pi_{yx}b(z)\|^2 < (1 + \delta/2)n$$

Hence the result follows from Theorem A. \square

Remark 2.6. By Kingman's subadditive ergodic theorem, the linear rate of escape

$$\lim_n \frac{1}{n} |X_n(\omega)|_G = \lim_n \frac{1}{n} \mathbb{E} |X_n|_G =: l_\mu$$

exists and is constant for a.e. $\omega \in (G, \mu)^\mathbb{N}$. Hence either of the conditions (1) and (2) in Corollary 2.5 implies that $l_\mu = 0$ and in particular that G is amenable ([16]).

Remark 2.7. It is known that $\mathbb{Z} \wr \mathbb{Z}$ does not satisfy property H_{FD} ([30, 5.4.1]). Shalom shows that any infinite amenable group with H_{FD} admits a virtual quotient to \mathbb{Z} ([30, 4.3.1]). By Corollary 2.5, any non-degenerate random walk on a group without virtual homomorphisms to \mathbb{Z} (or $\mathbb{Z} \wr \mathbb{Z}$) does not satisfy either of the conditions (1) or (2). It is apparently an open problem whether the wreath product $\mathbb{Z}^2 \wr (\mathbb{Z}/2\mathbb{Z})$ has property H_{FD} (see [30, 6.6]); the simple random walk on it does not satisfy either of the conditions (for "switch-walk-switch" random walks it follows from Dvoretzky–Erdős theorem ([7, 17]) that the number of distinct sites of a simple random walk on \mathbb{Z}^2 visited until the time n is asymptotically equivalent to $cn/\log(n)$, where $c > 0$ is a constant).

3. MORE ON THE PROPERTY H_{FD}

We elaborate on Corollary 2.5. It says G has property H_{FD} provided that (G, μ) satisfies the following property. We say a μ -random walk X_n is *cautious* if

$$\limsup_n \mathbb{P}(\max_{k=1, \dots, n} |X_k|_G < c\sqrt{n}) > 0$$

for every $c > 0$. We look at stability of this property under extension. Let N be a subgroup of G with a pseudo-length $|\cdot|_N$. We say N is *strictly exponentially distorted* in G if there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \log(|h|_N + 1) - C \leq |h|_G \leq C \log(|h|_N + 1) + C$$

for all $h \in N$. We will denote by $|\cdot|_{G/N}$ the pseudo-length induced by the compact generating neighborhood QN of e in G/N .

Proposition 3.1. *Let $N \triangleleft G$ be a closed normal subgroup which is strictly exponentially distorted, and let $\bar{\mu}$ be the push-out probability measure of μ to G/N . If $(G/N, \bar{\mu})$ is cautious, then so is (G, μ) and in particular G has Shalom's property H_{FD} .*

Proof. It suffices to show that there is a constant $D \geq 1$ with the following property (cf. [34, Lemma 3.4]). Let $s_i \in G$ be such that $|s_i|_G \leq 1$ and put $g_k := s_1 \cdots s_k \in G$ and $M_n := \max_{k=1, \dots, n} |g_k N|_{G/N}$. Then, one has $\max_{k=1, \dots, n} |g_k|_G \leq D(M_n + \log n + 1)$. To show such D exists, for each k , pick $h_k \in N$ such that $|g_k^{-1} h_k|_G = |g_k^{-1} N|_{G/N} \leq M_k$. Then, $|h_{k-1}^{-1} h_k|_G \leq 2M_k + 1 \leq 3M_k$ and so $|h_{k-1}^{-1} h_k|_N \leq \exp(4CM_k)$. Hence $|h_k|_N \leq n \exp(4CM_n)$ for all $k \leq n$ and so

$$\max_{k=1, \dots, n} |h_k|_N \leq C \log(2n \exp(4CM_n)) \leq (D-1)(M_n + \log n + 1)$$

for some constant $D \geq 1$. Since $|g_k^{-1} h_k|_G \leq M_n$, we are done. \square

Shalom ([30, Theorem 1.13]) has shown that polycyclic groups have property H_{FD} by invoking Delorme's theorem ([6]) that connected solvable Lie groups have the corresponding property, and asked if there is another proof of H_{FD} . It is plausible that all connected solvable groups are cautious. We note that in light of Osin's result ([27]) this problem reduces to the case for connected Lie groups with polynomial volume growth.

Corollary 3.2. *Let K be a non-archimedean local field and $\mathbb{Z}^d \curvearrowright K^n$ be a semi-simple linear action such that the semi-direct product $\mathbb{Z}^d \ltimes K^n$ is compactly generated. Then, $\mathbb{Z}^d \ltimes K^n$ has Shalom's property H_{FD} .*

Proof. Let ν_0 be the standard nearest neighborhood random walk on \mathbb{Z}^d and ν_1 be a uniform probability measure on the compact subgroup $\{x \in K : |x| \leq 1\}$. Since (\mathbb{Z}^d, ν_0) is cautious, for $\mu = \frac{1}{2}(\nu_0 + \nu_1^{\otimes n})$, the random walk $(\mathbb{Z}^d \ltimes K^n, \mu)$ is cautious. \square

4. HARMONIC COCYCLE $b_{\mu,U}$ CONSTRUCTED FROM DIFFERENCES OF SHIFTS OF μ^{*n}

In this section, we assume G is discrete (moreover finitely generated by Convention) and amenable, and we study a μ -harmonic 1-cocycle $b_{\mu,U}$, obtained as a limit of normalized $\mu^{*n} - g\mu^{*n}$.

We fix a non-principal ultrafilter U on \mathbb{N} and denote by \lim_U the corresponding ultralimit. Then, the ultrapower Hilbert space \mathcal{H}^U of a given Hilbert space \mathcal{H} is defined to be

$$\mathcal{H}^U := \ell_\infty(\mathbb{N}; \mathcal{H}) / \{(v_n)_{n=1}^\infty : \lim_U \|v_n\| = 0\}$$

with the inner product $\langle [v'_n]_n, [v_n]_n \rangle := \lim_U \langle v'_n, v_n \rangle$, where $[v_n]_n$ is the equivalence class of $(v_n)_n \in \ell_\infty(\mathbb{N}; \mathcal{H})$. An orthogonal representation π of G on \mathcal{H} gives rise to the ultrapower representation π^U on \mathcal{H}^U by $\pi_g^U [v_n]_n = [\pi_g v_n]_n$.

To avoid parity problems, we assume (in addition to Convention) that $\mu = \mu'^{*2}$, where μ' is some symmetric probability measure on G . We consider the left regular representation λ on $\ell_2(G)$ and put

$$\gamma(n) := \int \|\mu^{*n/2} - g\mu^{*n/2}\|_2^2 d\mu(g) = 2(\mu^{*n}(e) - \mu^{*n+1}(e))$$

(we omit writing λ and simply write g for λ_g) and

$$b_n(g) := \gamma(n)^{-1/2}(\mu^{*n/2} - g\mu^{*n/2}) \in \ell_2(G).$$

Since the sequence $\|b_n(g)\| \leq \|g\|_G \|b_n\|_Q$ is bounded for each g , the formula

$$b_\mu(g) := [b_n(g)]_n \in \ell_2(G)^U$$

defines a 1-cocycle (with respect to λ^U) such that $\|b_\mu\|_{L^2(\mu)} = \lim_U \|b_n\|_{L^2(\mu)} = 1$, where we have used the assumption that $\int |x|_G^2 d\mu(x) < \infty$ to interchange the limit and integration. We will show in Theorem C, that the 1-cocycle b_μ may depend on the choice of a non-principle ultrafilter U , and we will write b_μ^U instead of b_μ when we want to emphasize the role of U .

Recall that if G is a amenable group, then by Kesten's criterion ([20]) the probability to return to the origin of any symmetric measure is sub-exponential. Moreover, $\mu^{*n}(e)$ is decreasing and $\mu^{*n+1}(e)/\mu^{*n}(e) \nearrow 1$ (see Avez [1]). The following lemma pushes this one step further.

Lemma 4.1. *Let G be a discrete infinite amenable group and assume (in addition to Convention) that $\mu = \mu'^{*2}$ for some probability measure μ' on G . Then, one has $\lim_n \gamma(n+1)/\gamma(n) = 1$, where $\gamma(n) = 2(\mu^{*n}(e) - \mu^{*n+1}(e))$.*

Proof. Since $\gamma(n) = 2(\mu^{*n}(e) - \mu^{*n+1}(e)) = 2\langle (1 - \lambda(\mu))\lambda(\mu)^n \delta_e, \delta_e \rangle$ and $\lambda(\mu)$ is a positive contraction, the sequence γ is decreasing. Put

$$\delta(n) := \gamma(2n) + \gamma(2n+1) = 2(\mu^{2n}(e) - \mu^{2(n+1)}(e)).$$

Then $\delta(n)$ is also decreasing and moreover

$$\begin{aligned}\delta(n+1)^2 &= \left(\int \langle \mu^{*n} - g\mu^{*n}, \mu^{*n+2} - g\mu^{*n+2} \rangle d\mu^{*2}(g) \right)^2 \\ &\leq \left(\int \|\mu^{*n} - g\mu^{*n}\|^2 d\mu^{*2}(g) \right) \left(\int \|\mu^{*n+2} - g\mu^{*n+2}\|^2 d\mu^{*2}(g) \right) \\ &= \delta(n)\delta(n+2).\end{aligned}$$

We know that $\delta(n+1)/\delta(n)$ is increasing, let δ ($\delta \leq 1$) be the limit of this sequence. If $\delta < 1$, then $\delta(n) \leq C\delta^n$ and so $\mu^{*n}(e) = \frac{1}{2} \sum_{k=n}^{\infty} \gamma(k) \leq C'\delta^{n/2}$ (here $\lim_m \mu^{*m}(e) = 0$ since the support of μ generates G and since G is infinite), which is in contradiction with amenability of G . Thus $\delta = 1$, and since

$$\frac{\delta(n+1)}{\delta(n)} \leq \frac{\gamma(2n+2) + \gamma(2n+3)}{2\gamma(2n+1)} \leq \frac{1}{2} \frac{\gamma(2n+2)}{\gamma(2n+1)} + \frac{1}{2} \frac{\gamma(2n+3)}{\gamma(2n+2)} \leq 1,$$

one has $\lim_n \gamma(n+1)/\gamma(n) = 1$. \square

Corollary 4.2. 1) Let G be an amenable group. Then, $b_{\mu,U}$ is a normalized μ -harmonic 1-cocycle for any choice of a non-principal ultrafilter U .

2) Let G be a group which does not have Kazhdan's property (T). Then, there exist an orthogonal representation (π, \mathcal{H}) of G and a unit vector v such that $\langle \pi(\mu^{*n})v, v \rangle \rightarrow 0$ and $\langle \pi(\mu^{*n})v, v \rangle^{1/n} \rightarrow 1$. For any such (π, \mathcal{H}, v) ,

$$b_v(g) := [(2\langle \pi(\mu^{*n} - \mu^{*n+1})v, v \rangle)^{-1/2} \pi(\mu^{*n/2} - g\mu^{*n/2})v]_n \in \mathcal{H}^U$$

is a normalized μ -harmonic 1-cocycle.

Proof. 1) Lemma 4.1 implies that 1-cocycle $b_{\mu,U}$ is μ -harmonic, since

$$\| \int b_{\mu,U}(x) d\mu(x) \|^2 = \lim_U \frac{\|\mu^{*n/2} - \mu^{*(n/2+1)}\|_2^2}{\gamma(n)} = \lim_U \frac{\gamma(n) - \gamma(n+1)}{2\gamma(n)} = 0.$$

2) Observe that if π has approximately invariant vectors but no non-zero invariant vectors, then 1 is contained in the spectrum of the self-adjoint contraction $\pi(\mu)$ but is not an eigenvalue of it, that is to say, the spectral projections $\chi_{(1-1/m, 1)}(\pi(\mu))$ are non-zero. Hence there is a unit vector $v \in \mathcal{H}$ such that $\chi_{(1-1/m, 1)}(\pi(\mu))v \neq 0$ for all m . Any such v satisfies the desired property. Proof of Lemma 4.1 applies verbatim and implies again that $b_v(g)$ is a normalized harmonic cocycle. \square

Proof of Theorem B. By Theorem A we know that $\mathbb{E}[|\frac{\|c(X_m)\|^2}{m} - 1|^2] \rightarrow 0$ for any normalized μ -harmonic 1-cocycle c without non-zero finite-dimensional summands. We will show that in case G does not admit any non-zero μ -harmonic finite-dimensional 1-cocycle (which is the case when G is a finitely generated amenable group without virtually abelian infinite quotients), this convergence is uniform for normalized μ -harmonic 1-cocycles c on G . Indeed, we have seen in the proof of Theorem A that

$$\mathbb{E}[|\frac{\|c(X_m)\|^2}{m} - 1|^2] \leq \frac{4}{m^2} \left\langle \sum_{k=1}^{m-1} (m-k) T^{k-1} w, w \right\rangle + \frac{1}{m} \|c\|_Q^4 \mathbb{E}[|X_1|_G^4] \rightarrow 0$$

for every normalized μ -harmonic 1-cocycle c , where $T = \int (\pi \otimes \bar{\pi})_g d\mu(g)$ and $w = \int (c \otimes \bar{c})(g) d\mu(g)$. Note that $\|c\|_Q$ is uniformly bounded by Lemma 2.1. Therefore, it suffices to prove that $\lim_k \|T^k w\| = 0$ uniformly for c . Suppose that the latter is not the case: there are $\varepsilon > 0$, a subsequence $k_m \rightarrow \infty$, and normalized μ -harmonic 1-cocycles c_m such that $\|T_{m_m}^{k_m}\| \geq \varepsilon$ for all m . Fix a non-principal ultrafilter U and let c_U denote

the U -ultralimit cocycle of the sequence $(c_m)_m$, with the corresponding objects denoted by T_U and w_U . Then, c_U is a normalized μ -harmonic 1-cocycle. Moreover since t^{2k} is decreasing in k for any $t \in [-1, 1]$, one has for each k

$$\langle T_U^{2k} w_U, w_U \rangle = \lim_U \langle T_m^{2k} w_m, w_m \rangle \geq \lim_U \langle T_m^{2k} w_m, w_m \rangle \geq \varepsilon^2.$$

Let Q denote the spectral projection of T_U corresponding to eigenvalues $\{-1, +1\}$. Then,

$$\|Qw_U\|^2 = \lim_k \langle k^{-1}(1 + T^2 + T^4 + \dots + T^{2(k-1)})w_U, w_U \rangle \geq \varepsilon^2.$$

Since $T_U^2 Qw_U = Qw_U$, the vector Qw_U is invariant under $(\pi \otimes \bar{\pi})_g$ for all $g \in \text{supp } \mu^{*2}$. However since $G_0 := \langle \text{supp } \mu^{*2} \rangle$ has finite-index in G , it does not admit a non-zero μ^{*2} -harmonic 1-cocycle, which implies that $Qw_U = 0$ (as discussed in the proof of Theorem A). We have arrived at a contradiction.

It follows that if G satisfies the assumption of Theorem B, then $\mathbb{E}[|m^{-1}\|c(X_m)\|^2 - 1|^2] \rightarrow 0$ uniformly for normalized μ -harmonic 1-cocycles c . In particular,

$$\lim_m \limsup_n \mathbb{E}\left[\left|\frac{\|b_n(X_m)\|^2}{m} - 1\right|^2\right] = \limsup_m \lim_U \mathbb{E}\left[\left|\frac{\|b_{\mu,U}(X_m)\|^2}{m} - 1\right|^2\right] = 0.$$

(Note that $\limsup_n \lambda_n = \sup_U \lim_U \lambda_n$ for any bounded sequence λ_n .) Since $\lim_m \lim_n m(\mu^{*n}(e) - \mu^{*n+1}(e))/(\mu^{*n}(e) - \mu^{*n+m}(e)) = 1$ by Lemma 4.1, this completes the proof of Theorem B (modulo exchanging μ with μ^{*2}). \square

5. ℓ_p -THIN SUBGROUPS

5.1. Definitions. Take a finitely generated group G equipped with a probability measure μ , and ask again what information about its subgroups and quotient groups one can obtain by looking on the behavior the random walk (G, μ) . To ensure the existence of non-trivial quotients, we may search normal subgroups of G defined by convolutions of G . A more general question one can ask is what are possible (not necessarily normal subgroups) defined in such terms.

Definition 5.1. [ℓ_p -thin subgroups $H_{\mu,p,q}$]. Let G be a group generated by a finite set S , and μ be a probability measure on G . Fix some $p \geq 0$, $q \geq 1$ and a sequence n_i tending to ∞ . Let $\alpha(n)$ denotes the maximum of ℓ_p norm of $(\mu^{*n})^q - g(\mu^{*n})^p$, where the maximum is taken over $g \in S$. Consider $g \in G$ for which $\|(\mu^{*n_i})^q - g(\mu^{*n_i})^q\|_p / \alpha(n_i) \rightarrow 0$ as $i \rightarrow \infty$. By triangular inequality in ℓ_p such elements form a subgroup of G , which we call *the main ℓ_p -thin subgroup* and which we denote by H_{μ,p,q,n_i} (and $H_{\mu,p}$ for short, if n_i is specified and $q = 1$).

Now we define ℓ_p -thin subgroups associated an arbitrary function $\alpha(n)$. Consider g such that $\|(\mu^{*n})^q - g(\mu^{*n})^q\|_p / \alpha(n)$ tends to 0 as n tends to infinity. By triangular inequality in ℓ_p such elements form a subgroup of G , which we denote $H_{\mu,p,q,\alpha}$. We call this subgroup ℓ_p -thin subgroup associated to $\alpha(n)$.

It is clear that the scaling sequence $\alpha(n)$ depends of a finite generating set S up to multiplication by a constant only, and thus the definition of main ℓ_p -thin subgroups does not depend on the choice of S .

For $p = 0$ in the definition above we use the convention $0^0 = 0$; the ℓ_1 norm in this case is therefore the cardinality of the support $\mu^{*n} - g\mu^{*n}$.

In many situation the limit behavior of $(\mu^{*n})^p - g(\mu^{*n})^p$ does not depend on the subsequence of possible n 's. However, in some situation this quantity, and the corresponding ℓ_p -thin subgroups may depend on the choice of a subsequence, see Theorem C and Corollary 5.10.

Remark 5.2. If $p \geq 1$, it is known that a sequence $v_n \in \ell_1(G)$ is almost invariant in ℓ_1 with respect to the shift by some element $g \in G$ if and only if $v_n^{1/p}$ (which is clearly a sequence in $\ell_p(G)$) is almost invariant in ℓ_p with respect to the shift by g (see e.g. the proof of Theorem 8.3.2 in [29]). This implies that the main ℓ_p -thin subgroups satisfy $H_{p,1} = H_{1,p} = H_{p/q,q}$ for any $p, q \geq 1$ whenever $(\mu^{*n_i})^p$ does not admit a subsequence of almost invariant vectors in ℓ_1 . This happens for example for $p = 2$ if G non-amenable and for $p = 1$ if the Poisson boundary of (G, μ) is non-trivial, for all n_i ([19]). For $p = 0$ and some choice of n_i this happens for any group G of exponential growth.

It is possible that the statement of Remark 5.2 remains valid without the assumption of non-almost-invariance.

Instead $(\mu^{*n})^q$ in the Definition 5.1, one can consider more generally a sequence of functions f_i and consider the difference of corresponding shifted functions, as a function of g .

We have already remarked that for $p = 2, q = 1$, μ being equidistributed on a finite symmetric set of G , the values of $b_{\mu,U}$ are defined by the unmarked Cayley graph of G . In particular, for $p = 2, q = 1$ and μ being a measure equidistributed on a finite generating set S , the ℓ_p -thin subgroups can be described in terms of unmarked Cayley graph of (G, S) :

Remark 5.3. $p = 2, q = 1$, μ is symmetric measure on G . An element g belongs to the subgroup $H_{\mu,2,1,\alpha}$ if and only if

$$(\mu^{2n}(e) - \mu^{*2n}(g))/\alpha^2(n) \rightarrow 0$$

as $n \rightarrow 0$. In particular, if μ is equidistributed on a finite symmetric generating set S , subgroups $H_{\mu,2,\alpha}$ are defined by unmarked Cayley graph of (G, S) .

Proof. Observe that $|g\mu^{*n} - \mu^{*n}|_2^2 = |\mu^{*n}|_2^2 + |g\mu^{*n}|_2^2 - 2\langle \mu^{*n}, g\mu^{*n} \rangle = 2|\mu^{*n}|_2^2 - 2\langle \mu^{*n}, g\mu^{*n} \rangle = 2(\sum_{x \in G} (\mu^{*n}(x))^2 - \sum_{x \in G} \mu^{*n}(x)\mu^{*n}(gx))$. Since μ is symmetric, this is equal to $2(\sum_{x \in G} \mu^{*n}(x)\mu^{*n}(x^{-1}) - \sum_{x \in G} \mu^{*n}(gx)\mu^{*n}(x^{-1})) = 2(\mu^{2n}(e) - \mu^{*2n}(g))$. If μ is equidistributed on a finite symmetric generating set S , observe that $\mu^{*2n}(e)$ and $\mu^{*2n}(g)$ are defined by the unmarked Cayley graph of (G, S) and the vertex in this Cayley graph corresponding to g .

Remark 5.4. In a particular case when $q = 1, p = 2$ and G is non-amenable, the main ℓ_2 -thin subgroup in 5.1 coincides with the group, studied by Elder and Rogers in [8]. However, if $q = 1, p = 2$ and G is amenable, the group defined in the above cited paper coincides with G , while the main ℓ_2 -thin subgroup $H_{\mu,p}$ is never equal to G unless G is a group consisting of one element.

Lemma 5.5. 1) [Direct products, $q = 0, p \geq 1$] Let G be a direct product of A of subexponential growth and B of exponential growth, and let $\mu = \mu_A \times \mu_B$. Then there exists a subsequence n_i such that the corresponding main ℓ_0 -thin subgroup $H_{\mu,0,p}(G) = H_{\mu,p,0}(G)$ contains A . Moreover, for $q = 0$ and $p \geq 1$ and n_i as above the cocycles $b_{n_i,\mu}^{p,q}$ do not depend on p and $b_{n_i,\mu}^{p,q} = b_{n_i,\mu_B}^{p,q}$.

2) [Direct product, $p = q = 1$] Let G be a direct product of a group A and B ; let μ_A, μ_B be non-degenerate measures on A and B such that the Poisson boundary of a random walk (A, μ_A) is trivial and Poisson boundary of (B, μ_B) is non-trivial. Put $\mu = \mu_A \times \mu_B$. Then for any choice of n_i the main ℓ_1 -thin subgroup $H_{\mu,1,1}(G)$ contains A . Moreover, for any n_i it holds $b_{n_i,\mu}^{1,1} = b_{n_i,\mu_B}^{1,1}$.

3) [Direct products, $q = 1, p = 2$] Let G be a direct product of an amenable A and non-amenable group B , $\mu = \mu_A \times \mu_B$. Then for any n_i , the main ℓ_2 -thin subgroup $H_{\mu,1,2}(G) = H_{\mu,2,1}(G)$ contains A . Moreover, for any n_i it holds $b_{n_i,\mu}^{2,1} = b_{n_i,\mu_B}^{2,1}$.

Proof. First we prove the claims of 1), 2), 3) about ℓ_p -thin subgroups. Observe that since B is of exponential growth, for any finite set S there exists $v > 1$ such that $v_{G,S}(n) \geq v^n$ for all n . This implies that for each finite generating set S_B of B and each $C_1 < 1$ there exists $C_2 > 0$ such that for all n at least $C_1 n$ among balls of radius $i = 1, \dots, n$ have boundary greater than $C_2 v_{B,S_B}(i)$. (Indeed, otherwise $v_{B,S_B}(n) \leq R_B^{n(1-C_1)}(1+C_2)^{C_1 n}$, where R_B denotes the cardinality of B, S_B , and taking C_1 close to 1 and C_2 close to 0 we would get a contradiction).

Since A is of subexponential growth, for each C and any $\epsilon_1, \epsilon_2 > 0$ at least $(1 - \epsilon_2)n$ among the balls of radius $i = 1, \dots, n$ have boundary at most $\epsilon_2 v_A(i)$.

Consider a generating set $S = S_A \times S_B$, where S_A, S_B are generating sets of A, B respectively. We have $B_S(i) = B_{S_A}(i) \times B_{S_B}(i)$. Here $B_{G,S}(i)$ denotes the ball of radius i in G, S . Observe also that for $s \in S_A$ it holds $sB_{G,S}(i) \setminus B_{G,S}(i) = s(B_{A,S_A}(i) \times B_{B,S_B}(i)) \setminus (B_{A,S_A}(i) \times B_{B,S_B}(i)) = (sB_{A,S_A}(i) \setminus B_{A,S_A}(i)) \times B_{B,S_B}(i)$, and the cardinality of this set is at most $2(v_{A,S_A}(i) - v_{A,S_A}(i-1))v_{B,S_B}(i)$, and with the same argument the cardinality of $sB_{G,S}(i) \setminus B_{G,S}(i)$, for $s \in S_B$ is at least $2/|S_B|(v_{B,S_B}(i) - v_{B,S_B}(i-1))v_{A,S_A}(i)$ for some $s \in S_B$. Note that if n_i is such that

$$\frac{v_{A,S_A}(n_i) - v_{A,S_A}(n_i - 1)}{v_{A,S_A}(n_i)} \frac{v_{B,S_B}(n_i)}{v_{B,S_B}(n_i) - v_{B,S_B}(n_i - 1)}$$

tends to 0 as i tends to infinity, then the ℓ_0 -thin subgroup with respect to a subsequence n_i contains all $s \in S_A$. Therefore, in this case this subgroup contains A . By Remark 5.2 we know that under assumption of 1) it holds $H_{\mu,0,p}(G) = H_{\mu,p,0}(G)$.

2) We recall that $\mu^{*n} = \mu_A^{*n} \mu_B^{*n}$. Take $a \in A$. Observe that $a\mu^{*n} - \mu^{*n} = (\mu_A^{*n} - a\mu_A^{*n})\mu^{*n}(B)$. It holds therefore $|(\mu^{*n} - a\mu^{*n})|_1 = |(\mu_A^{*n} - a\mu_A^{*n})|_1$. Since the non-degenerate walk (A, μ_A) has trivial Poisson-Furstenberg boundary, for any $a \in A$ it holds $|(\mu_A^{*n} - a\mu_A^{*n})|_1 \rightarrow 0$ as n tends to ∞ , and therefore $|(\mu^{*n} - a\mu^{*n})|_1 \rightarrow 0$ as n tends to ∞ (see Kaimanovich Vershik [19]). The above mentioned characterization also shows that since the Poisson boundary of (B, μ_B) is non-trivial, there exists $b \in B$ such that $|(\mu_B^{*n} - b\mu_B^{*n})|_1 \geq c > 0$, and hence $|(\mu^{*n} - b\mu^{*n})|_1 \geq c > 0$ for some positive constant c and all n .

3) For $g = (g_1, g_2, g_1 \in A, g_2 \in B, \mu^{*n}(g_1, g_2) = \mu^{*n}(g_1)\mu^{*n}(g_2)$. For $h \in A$, $\mu^{*n}(h(g_1, g_2))/\mu^{*n}(g_1, g_2) = \mu_A^{*n}h/\mu_A^{*n}(g_1) \rightarrow 1$ as $n \rightarrow \infty$, by [1] since A is amenable [1]. Analogously, $\mu^{*n}(h(g_1, g_2))/\mu^{*n}(g_1, g_2) = \mu^{*n}h(g_1, g_2)/\mu^{*n}(g_1, g_2) = \mu_B^{*n}h(g_2)/\mu_B^{*n}h(g_2) \rightarrow C_h$, where $C_h \neq 1$ for some h among generators of B , since B is non-amenable [1]. This implies that the scaling sequence $\alpha(n)$ is equivalent up to multiplicative constant to $\mu^{*n}(e) = \mu_A^{*n}(e)\mu_B^{*n}(e)$. Using Remark 5.3 we conclude that for all $s \in S_A$ $\|s\mu^{*n} - \mu^{*n}\|_2/\alpha(n) \rightarrow 0$, and hence any $s \in A$, s belongs to the ℓ_2 thin subgroup for $q = 1, p = 2$. By Remark 5.2 we know that under assumption of 3) it holds $H_{\mu,1,2}(G) = H_{\mu,2,1}(G)$.

Now to prove the claims about the cocycles, take $g = (a, b) \in A \times B$, put $g' = (e, b)$ and $g'' = (a, e)$. It holds $g = g'g''$. Under the assumption on p and q in 1), 2), 3) observe that

$$\begin{aligned} \|(\mu^{*n_i})^q - g'(\mu^{*n_i})^q\|_p - \|(\mu^{*n_i})^q - g''(\mu^{*n_i})^q\|_p &\leq \|(\mu^{*n_i})^q - g(\mu^{*n_i})^q\|_p \leq \\ &\leq \|(\mu^{*n_i})^q - g'(\mu^{*n_i})^q\|_p + \|(\mu^{*n_i})^q - g''(\mu^{*n_i})^q\|_p \end{aligned}$$

and that $\|(\mu^{*n_i})^q - g'(\mu^{*n_i})^q\|_p = \|(\mu_B^{*n_i})^q - g'(\mu_A^{*n_i})^q\|_p \|\mu_A^{*n_i}\|_p$. This allows us to use 1) of Remark 5.6 and completes the proof of the Lemma.

Remark 5.6. $G = A \times B$, $\mu = \mu_A \times \mu_B$, $S = S_A \times S_B$, S_A and S_B are finite generating sets of A and B .

1) Let $\alpha_G^{p,q}(n)$ be the maximal ℓ_p norm of $(\mu^{*n})^q - s(\mu^{*n})^q$, where the maximum is taken over $s \in S$; and let $\alpha_A^{p,q}(n)$ be the maximal ℓ_p norm of $(\mu_A^{*n})^q - s(\mu_A^{*n})^q$, the maximum is over $s \in S_A$ and $\alpha_B^{p,q}(n)$ is defined analogously. Let $\tilde{\theta}_\mu(n)$ be equal to $\alpha_G^{p,q}(n)$ divided by the ℓ_p norm of $(\mu^{*n})^q$. If $\tilde{\theta}_A^{p,q}(n_i)/\tilde{\theta}_B^{p,q}(n_i)$ tends to zero for some sequence n_i and U is a non-principal ultrafilter such that $U(\{n_i\}) = 1$, then $b_{\mu,U}^{p,q} = b_{\mu_B,U}^{p,q}$.

2) Take $q = 1$, $p = 2$. Put $\theta(n) := (\mu^{*2n} - \mu^{*2n+1})/\mu^{*2n}$. Then $\theta(n) = \tilde{\theta}_\mu(n)^2$. In particular, if $\theta_A(n_i)/\theta_B(n_i)$ tends to zero and U is a non-principal ultrafilter such that $U(\{n_i\}) = 1$, then the corresponding harmonic cocycle is defined by that of B , that is $b_{\mu,U} = b_{\mu_B,U}$.

Remark 5.7. The fact that $A \times B$, A is of subexponential growth, B is of exponential growth, satisfy the claim of 1), Lemma 5.5 not only for some sequence n_i but for all sequences can be shown to be equivalent to a positive answer to both following questions A): is it true that no subset of balls is a Foelner sequence in A ? B): Is it true that all balls form a Foelner sequence in A ?

To our knowledge, it is not known whether to answer to A) is positive for all groups of exponential growth (this question is mentioned e.g. in [33]), and whether the answer to B) is positive for all groups of subexponential growth.

Example 5.8 (Dependence of ℓ_p -thin subgroups on p). Let $G = F_m \times \mathbb{Z}^d \wr A$, where $m \geq 2$, $d \geq 3$ and A is a finite group containing at least two elements. Let μ be a non-degenerate symmetric finitely supported measure. Then ℓ_2 -thin subgroup is not equal to ℓ_1 -thin subgroup.

Proof. Observe that the ℓ_2 -thin subgroup $H_{\mu,1,2} = H_{\mu,2,1}$ contains $\mathbb{Z}^d \wr A$ by 3) of Lemma 5.5 (in fact, it is equal to $\mathbb{Z}^d \wr A$), while there exists $g \in \mathbb{Z}^d \wr A$ which does not belong to ℓ_1 -thin subgroup since the Poisson boundary of $\mathbb{Z}^d \wr A$ is non-trivial.

Remark 5.9. Let $G = C \wr A$, where C is an infinite group of at least cubic growth and A is finite. Let μ be a symmetric finitely supported "switch-walk-switch" measure on G . Consider the subgroup $H_\mu^{\ell_1}$ of G containing $h \in H : \sum_g |\mu^{*n}(hg) - \mu^{*n}(g)| \rightarrow 0$. One can show that $H_\mu^{\ell_1}$ is a finite subgroup of G . One can also show that for any integer $k \geq 0$ there exists μ as above such that $H_\mu^{\ell_1}$ is isomorphic to A^m . In particular, this main ℓ_1 thin subgroup $H_\mu^{\ell_1}$ depends on the choice of a finitely supported symmetric measure μ and this subgroup is not normal unless it contains one element

Proof of Theorem C. Assume $d = 2$ (the general case $d \geq 2$ is analogous).

We construct G_1 and G_2 as piecewise-automatic groups with returns of automata τ_1 , τ_2 , where $\tau_1, \tau_2 : A \times X \rightarrow A$, the group generated by (A, τ_1) is of intermediate growth, $\tau_2 : A \times X \rightarrow A$, the group H_2 generated by (A, τ_2) is non-amenable, and the action of A , considered as generators of H , is contracting for the action of τ_1 for each brach of the rooted tree (see [9]).

More precisely, we chose automata τ_1 and τ_2 with the following properties: τ_2 is a finite state automaton, containing e, a, b, c, d as its states, such that e acts trivially and a, b, c, d generate the free product $\mathbb{Z}/2\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z})$ in the group generated by τ_2 .

If the states of τ_2 are e, a, b, c, d and the alphabet is $0, 1$, we take as τ_1 the standard finite state automaton for the first Gogorichuk group ($A = \{e, a, b, c, d\}$, $X = \{0, 1\}$). In this case we can take as G_1 and G_2 either piecewise-automatic group or a piece-wise automatic group with returns defined by τ_1, τ_2 and $t_i, T_i, i \geq 1, T_{i-1} < t_i < T_i$. We do not know if

τ_2 as above exists, and therefore we consider as in [9] an automaton τ_2 with the space of states possibly larger than e, a, b, c, d (such automata exist by the result of Olijnyk [26], that shows that any free product of finite groups imbeds in a group generated by a finite state automaton), and we take as τ_1 the standard finite state automaton for the first Grigorchuk group, (extended to some larger alphabet than 0 and 1 if the alphabeth of τ_2 contains more than two letters) and consider the corresponding piecewise automatic group with returns $G_{\tau_1, \tau_2}(t_i, T_i)$.

To construct G_1 and G_2 , we fix τ_1, τ_2 and construct sequences t_i^1, T_i^1 and t_i^2, T_i^2 ($T_{i-1}^1 < t_i^1 < T_i^1$, $T_{i-1}^2 < t_i^2 < T_i^2$) by a simultaneous inductive procedure and we put $G_1 = G_{\tau_1, \tau_2}(t_i^1, T_i^1)$ and $G_2 = G_{\tau_1, \tau_2}(t_i^2, T_i^2)$.

We need the following properties of piece-wise automatic group with returns $G_{\tau_1, \tau_2}(t_i, T_i)$ (see proof of Proposition 1 in [9]). There exist $\Psi : \mathbb{N} \rightarrow \mathbb{N}$ and for each i there exist "comparison groups" $\mathcal{A}(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$ and $\mathcal{B}(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$, such that the following holds for all non-decreasing sequences t_i, T_i :

- (1) all groups $\mathcal{A}(t_1, T_1, t_2, T_2, \dots, T_{i-1})$ have a finite index subgroup which imbeds as a subgroup in a finite direct power of the first Grigorchuk group G_1 (generated by (A, τ_1)),
- (2) all groups $\mathcal{B}(t_1, T_1, t_2, T_2, \dots, T_{i-1}, t_i)$ have a finite index subgroup which admits a surjective homomorphism to the group, generated by the automaton (A, τ_2) ,
- (3) the balls of radius $\Psi(t_i)$ in $G(t_1, t_2, \dots, T_1, T_2, \dots)$ and $\mathcal{A}(t_1, T_1, t_2, T_2, \dots, t_{i-1}, T_{i-1})$ coincide,
- (4) the balls of radius $\Psi(T_i)$ in $G(t_1, t_2, \dots, T_1, T_2, \dots)$ and $\mathcal{B}(t_1, T_1, t_2, T_2, \dots, T_{i-1}, t_i)$ coincide.

Let G, S_G, H, S_H be finitely generated groups such that the balls of radius $R + C$ in the marked Cayley graphs of G, S_G, H, S_H coincide. Let μ_H and μ_G are measures which are equal after the identifications of these and such that $l_G(s) \leq C$ for any s in the support of μ_G . Observe that for any $n \leq R$ the scaling functions in the definition of ℓ_p -thin subgroups are equal : $\alpha_{G, \mu_G, p}(n) = \alpha_{H, \mu_H, p}(n)$, $\alpha'_{G, \mu_G, p}(n) = \alpha'_{H, \mu_H, p}(n)$ and for each g in the ball of radius C in the Cayley graph of (G, S_G) ℓ_p norms of $g(\mu_G^{*n})^q - (\mu_G^{*n})^q$ are equal to the ℓ_p norm of $h(\mu_H^{*n})^q - (\mu_H^{*n})^q$ for h being the corresponding element in the ball of radius C of (H, S_H) .

Suppose that we have chosen already $t_1^1, T_1^1, t_2^1, T_2^1, \dots, T_{i-1}^1$ and $t_1^2, T_1^2, t_2^2, T_2^2, \dots, t_i^2$. For any $\epsilon > 0$ there exist M_i such that for all $M'_i > M_i$ there exists M_i^* with the following property. For any $t_i^1 > M_i^*$ and $T_i^2 > M_i^*$ and any $n : M_i < n < M'_i$ the ratio of ℓ_p norms $s_1(\mu^{*n})^q - (\mu^{*n})^q$ and $s_2(\mu^{*n})^q - (\mu^{*n})^q$ in $G = G_1 \times G_2$ is smaller than ϵ for all $s \in S_1$ and some $s \in S_2$.

To prove this, we combine the observation about Cayley graphs above with the claims 1), 2), 3) of Lemma 5.5, for $\mathcal{A} = \mathcal{A}(t_1^1, T_1^1, t_2^1, T_2^1, \dots, T_{i-1}^1)$, $\mathcal{B} = \mathcal{B}(t_1^2, T_1^2, t_2^2, T_2^2, \dots, t_i^2)$. The group A is of intermediate growth and hence this group is amenable and finitely supported random walks have trivial boundary, B has a finite index subgroup subjecting to a non-amenable group, and hence non-amenable.

Now suppose that we have chosen already $t_1^1, T_1^1, t_2^1, T_2^1, \dots, t_i^1$ and $t_1^2, T_1^2, t_2^2, T_2^2, \dots, t_i^2, T_i^2$. For any $\epsilon > 0$ there exist N_i such that for all $N'_i > N_i$ there exists N_i^* with the following property. For any $T_i^1 > N_i^*$ and $t_{i+1}^2 > M_i^*$ and any $n : N_i < n < N'_i$ the ratio of ℓ_p norms of $s_2(\mu^{*n})^q - (\mu^{*n})^q$ and $s_1(\mu^{*n})^q - (\mu^{*n})^q$ in $G = G_1 \times G_2$ is smaller than ϵ for all $s_2 \in S_2$ and some $s_1 \in S_1$.

This implies that for some choice of t_i^1, T_i^1 and t_i^2, T_i^2 there exist sequences n_i, m_i tending to infinity, such that the following holds. The ratio of ℓ_p norms of $s_1(\mu^{*n_i})^q -$

$(\mu^{*n_i})^q$ and the scaling sequence $\alpha(n_i)$ tend to 0 for all $s_1 \in S_1$. This implies that all $s_1 \in S_1$, as well as all $g \in G_1$ belong to the main ℓ_p thin subgroup $H_{\mu,p,q}$, corresponding to n_i . The ratio of ℓ_p norms of $s_2(\mu^{*m_i})^q - (\mu^{*m_i})^q$ and the scaling sequence $\alpha(m_i)$ tend to 0 for all $s_2 \in S_2$. This implies that all $s_2 \in S_2$, as well as all $g \in G_2$ belong to the main ℓ_p thin subgroup $H_{\mu,p,q}$, corresponding to m_i . Using 1), 2), 3) of Lemma 5.5 we also observe that $f_{\mu,n_i}^{p,q}$ is equal to $f_{\mu_2,n_i,p,q}$ and that $f_{\mu,m_i,p,q}$ is equal to $f_{\mu_1,m_i}^{p,q}$.

Corollary 5.10. *Let G_i, μ_i be as in the formulation of Theorem C. Take $q = 0, 1$ or 2 and $p = 1$ or 2 . For each $j : 1 \leq j \leq D$ there exists $n_{i,j}$ such that for all the main ℓ_p -thin subgroup $H_{\mu,p,q}$ of G with respect to $n_i = n_{i,j}$ contains $\prod_{k:k \neq j} G_k$. In particular, there exist at least D not equal ℓ_p -thin subgroups.*

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